L^∞-uniqueness of Schrödinger operators restricted in an open domain*

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revised version 20 January 2008

Abstract

Consider the Schrödinger operator $\mathcal{A} = -\frac{\Delta}{2} + V$ acting on space $C_0^\infty(D)$, where D is an open domain in $\mathbb{R}^d$. The main purpose of this paper is to present the $L^\infty(D, dx)$-uniqueness for Schrödinger operators which is equivalent to the $L^1(D, dx)$-uniqueness of weak solutions of the heat diffusion equation associated to the operator $\mathcal{A}$.

Key Words: C_0-semigroups; L^∞-uniqueness of Schrödinger operators; L^1-uniqueness of the heat diffusion equation.

2000 AMS Subject Classification Primary: 47D03, 47F05 Secondary: 60J60

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*This work is partially supported by *Yangtze Research Programme*, Wuhan University, China, and the *Town Council* of Hunedoara, Romania.

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1 Preliminaries

Let D be an open domain in $\mathbb{R}^d$ with its boundary ∂D. We denote by $C_0^\infty(D)$ the space of all infinitely differentiable real functions on D with compact support. Consider the *Schrödinger operator* $\mathcal{A} = -\frac{\Delta}{2} + V$ acting on space $C_0^\infty(D)$, where Δ is the Laplace operator and $V : \mathbb{R}^d \longrightarrow \mathbb{R}$ is a Borel measurable potential.

The *essential self-adjointness* of Schrödinger operator in $L^2\left(\mathbb{R}^d, dx\right)$, equivalent to the unique solvability of Schrödinger equation in $L^2\left(\mathbb{R}^d, dx\right)$, has been studied by Kato [**Ka'84**], Reed and Simon [**RS'75**], Simon [**Si'82**] and others because of its importance in Quantum Mechanics. In the case where V is bounded, it is not difficult to prove that $\left(\mathcal{A}, C_0^\infty(\mathbb{R}^d)\right)$ is essentially self-adjoint in $L^2(\mathbb{R}^d, dx)$. But in almost all interesting situations in quantum physics, the potential V is unbounded. In this situation we need to consider the *Kato class*, used first by Schechter [**Sch'71**] and Kato [**Ka'72**]. A real valued measurable function V is said to be in the *Kato class* $\mathcal{K}^d$ on $\mathbb{R}^d$ if

$$\lim_{\delta \searrow 0} \sup_{x \in \mathbb{R}^d} \int\limits_{|x-y| \leq \delta} |g(x-y)V(y)|\, dy = 0$$

where

$$g(x) = \begin{cases} \frac{1}{|x|^{d-2}} & , \quad \text{if } d \geq 3 \\ \ln \frac{1}{|x|} & , \quad \text{if } d = 2 \\ 1 & , \quad \text{if } d = 1. \end{cases}$$

If $V \in L^2_{loc}\left(\mathbb{R}^d, dx\right)$ is such that V^- belongs to the Kato class on $\mathbb{R}^d$, it is well known that the Schrödinger operator $(\mathcal{A}, C_0^\infty(\mathbb{R}^d))$ is essentially self-adjoint and the unique solution in L^2 of the heat equation is given by the famous *Feynmann-Kac semigroup*

$\{P_t^V\}_{t\geq 0}$

$$P_t^V f(x) := \mathbb{E}^x f(B_t) exp\left(-\int\limits_0^t V(B_s)\,ds\right)$$

where f is a nonnegative measurable function, $(B_t)_{t\geq 0}$ is the Brownian Motion in $\mathbb{R}^d$ defined on some filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d}\right)$ with $\mathbb{P}_x(B_0 = x) = 1$ for any initial point $x \in \mathbb{R}^d$ and $\mathbb{E}^x$ means the expectation with respect to $\mathbb{P}_x$.

In the case where D is a strict sub-domain, sharp results are known only when $d = 1$ or, in the multidimensional case, only in some special situations.

Consequently of an intuitive probabilistic interpretation of uniqueness, WU [**Wu'98**] introduced and studied the uniqueness of Schrödinger operators in $L^1(D, dx)$. On say that $(\mathcal{A}, C_0^\infty(D))$ is $L^1(D, dx)$-unique if $\mathcal{A}$ is closable and its closure is the generator of some C_0-semigroup on $L^1(D, dx)$. This uniqueness notion was also studied in ARENDT [**Ar'86**], EBERLE [**Eb'97**], DJELLOUT [**Dj'97**], RÖCKNER [**Rö'98**], WU [**Wu'98**] and [**Wu'99**] and others in the Banach spaces setting.

2 $L^\infty(D, dx)$-uniqueness of Schrödinger operators

Our purpose is to study the $L^\infty(D, dx)$-uniqueness of the Schrödinger operator $(\mathcal{A}, C_0^\infty(D))$ in the case where D is a strict sub-domain on $\mathbb{R}^d$. But how we can define the uniqueness in $L^\infty(D, dx)$? One can prove rather easely that *the killed Feynmann-Kac* semigroup $\left\{P_t^{D,V}\right\}_{t\geq 0}$

$$P_t^{D,V} f(x) := \mathbb{E}^x 1_{[t<\tau_D]} f(B_t) exp\left(-\int\limits_0^t V(B_s)\,ds\right)$$

where $\tau_D := inf\{t > 0 \ :\ B_t \notin D\}$ is the first exiting time of D, is a semigroup of bounded operators on $L^p(D, dx)$ for any $1 \leq p \leq \infty$, which is strongly continuous for

$1 \leq p < \infty$, but never strongly continuous in $(L^{\infty}(D, dx), \| \,.\, \|_{\infty})$. Moreover, a well known result of LOTZ [**Lo'86**, Theorem 3.6, p. 57] says that the generator of any strongly continuous semigroup on $(L^{\infty}(D, dx), \| \,.\, \|_{\infty})$ must be bounded.

To obtain a correct definition of $L^{\infty}(D, dx)$-uniqueness, we should introduce a weaker topology of $L^{\infty}(D, dx)$ such that $\left\{ P_t^{D,V} \right\}_{t \geq 0}$ becomes a strongly continuous semigroup with respect to this new topology. Remark that the natural topology for studying C_0-semigroups on $L^{\infty}(D, dx)$ used first by WU and ZHANG [**WZ'06**] is *the topology of uniform convergence on compact subsets of* $L^1(D, dx)$, denoted by $\mathcal{C}(L^{\infty}, L^1)$. More precisely, if we denote

$$\langle f, g \rangle := \int_D f(x) g(x) dx$$

for all $f \in L^1(D, dx)$ and $g \in Ł^{\infty}(D, dx)$, then for an arbitrary point $g_0 \in L^{\infty}(D, dx)$, a basis of neighborhoods with respect to $\mathcal{C}(L^{\infty}, L^1)$ is given by

$$N(g_0; K, \varepsilon) := \left\{ g \in L^{\infty}(D, dx) \;:\; \sup_{f \in K} |\langle f, g \rangle - \langle f, g_0 \rangle| < \varepsilon \right\}$$

where K runs over all compact subsets of $L^1(D, dx)$ and $\varepsilon > 0$.

Remark that $(L^{\infty}(D, dx), \mathcal{C}(L^{\infty}, L^1))$ is a locally convex space and if $\{T(t)\}_{t \geq 0}$ is a C_0-semigroup on $L^1(D, dx)$ with generator $\mathcal{L}$, by [**WZ'06**, Tneorem 1.4, p. 564] it follows that $\{T^*(t)\}_{t \geq 0}$ is a C_0-semigroup on $(L^{\infty}(D, dx), \mathcal{C}(L^{\infty}, L^1))$ with generator $\mathcal{L}^*$.

Now we can introduce the uniqueness notion in $L^{\infty}(D, dx)$. Let $\mathbf{A}$ be a linear operator on $L^{\infty}(D, dx)$ with domain $\mathcal{D}$ wich is assumed to be dense in $L^{\infty}(D, dx)$ with respect to the topology $\mathcal{C}(L^{\infty}, L^1)$.

Definition 2.1. *The operator* $\mathbf{A}$ *is said to be a pre-generator on* $L^{\infty}(D, dx)$ *if there exists some* C_0*-semigroup on* $(L^{\infty}(D, dx), \mathcal{C}(L^{\infty}, L^1))$ *such that its generator* $\mathcal{L}$ *extends*

A. *We say that* **A** *is* $L^\infty(D,dx)$*-unique if* **A** *is closable and its closure with respect to the topology* $\mathcal{C}(L^\infty, L^1)$ *is the generator of some* C_0*-semigroup on* $(L^\infty(D,dx), \mathcal{C}(L^\infty, L^1))$.

The main result of this paper is

Theorem 2.2. *Let* $V \in L^\infty_{loc}(D,dx)$ *such that* $V^- \in \mathcal{K}^d$*. Then the Schrödinger operator* $(\mathcal{A}, C_0^\infty(D))$ *is* $(L^\infty(D,dx), \mathcal{C}(L^\infty, L^1))$*-unique.*

Proof. First, we must remark that the existence assumption of pre-generator in [**WZ'06**, Theorem 2.1, p. 570] is satisfied. Indeed, if consider the killed Feynman-Kac semigroup $\left\{P_t^{D,V}\right\}_{t\geq 0}$ on $L^\infty(D,dx)$ and for any $p \in [1,\infty]$ we define

$$\left\|P_t^{D,V}\right\|_p := \sup_{\substack{f\geq 0 \\ \|f\|_p\leq 1}} \left\|P_t^{D,V} f\right\|_p,$$

next lemma show that $\mathcal{A}$ is a pre-generator on $(L^\infty(D,dx), \mathcal{C}(L^\infty, L^1))$, i.e. $\mathcal{A}$ is contained in the generator $\mathcal{L}^{D,V}_{(\infty)}$ of the killed Feynmann-Kac semigroup $\left\{P_t^{D,V}\right\}_{t\geq 0}$.

Lemma 2.3. *Let* $V \in L^\infty_{loc}(D,dx)$ *such that* $V^- \in \mathcal{K}^d$ *and let* $\left\{P_t^{D,V}\right\}_{t\geq 0}$ *be the killed Feynman-Kac semigroup on* $L^\infty(D,dx)$*. If* $\left\|P_t^{D,V}\right\|_\infty$ *is bounded over the compact intervals, then* $\left\{P_t^{D,V}\right\}_{t\geq 0}$ *is a* C_0*-semigroup on* $(L^\infty(D,dx), \mathcal{C}(L^\infty, L^1))$ *and its generator* $\mathcal{L}^{D,V}_{(\infty)}$ *is an extension of* $(\mathcal{A}, C_0^\infty(D))$.

Proof. The proof is close to that of [**Wu'98**, Lemma 2.3, p. 288]. Let $\left\{P_t^{D,V}\right\}_{t\geq 0}$ be the killed Feynman-Kac semigroup on $L^\infty(D,dx)$. Remark that

$$\left|P_t^{D,V} f(x)\right| \leq P_t^{D,V}|f|(x) \leq P_t^{D,-V^-}|f|(x) \leq P_t^{-V^-}|f|(x)$$

from where we deduce that

$$\sup_{0\leq t\leq 1}\left\|P_t^{D,V}\right\|_\infty \leq \sup_{0\leq t\leq 1}\left\|P_t^{-V^-}\right\|_\infty < \infty$$

since $\left\|P_t^{-V^-}\right\|_\infty$ is uniformly bounded by the assumption that $V^- \in \mathcal{K}^d$ (see [**AS'82**]). Since $\left\|P_t^{D,V}\right\|_1 = \left\|P_t^{D,V}\right\|_\infty$ is bounded for t in compact intervals of $[0,\infty)$, using [**Wu'01**, Lemma 2.3, p. 59] it follows that $\left\{P_t^{D,V}\right\}_{t\geq 0}$ is a C_0-semigroup on $L^1(D,dx)$. By [**WZ'06**, Theorem 1.4, p. 564] we find that $\left\{P_t^{D,V}\right\}_{t\geq 0}$ is a C_0-semigroup on $L^\infty(D,dx)$ with respect to the topology $\mathcal{C}(L^\infty,L^1)$. We have only to show that its generator $\mathcal{L}_{(\infty)}^{D,V}$ is an extension of $(\mathcal{A}, C_0^\infty(D))$.

Step 1: the case $V \geq 0$. For $n \in \mathbb{N}$ we consider $V_n := V \wedge n$. By a theorem of bounded perturbation (see [**Da'80**, Theorem 3.1, p. 68]) it follows that

$$\mathcal{A}_n = -\frac{\Delta}{2} + V_n$$

is the generator of a C_0-semigroup $\left\{P_t^{D,V_n}\right\}_{t\geq 0}$ on $(L^\infty(D,dx), \mathcal{C}\left(L^\infty,L^1\right))$. So for any $f \in \mathcal{C}_0^\infty(D)$ we have

$$P_t^{D,V_n} f - f = \int_0^t P_s^{D,V_n} \mathcal{A}_n f \, ds \quad , \quad \forall t \geq 0.$$

Letting $n \to \infty$, we have pointwisely on D:

$$P_t^{D,V_n} f \to P_t^{D,V} f$$

and

$$P_t^{D,V_n} \mathcal{A}_n f \to P_t^{D,V} \mathcal{A} f \quad .$$

Moreover, for any $x \in D$ we have:

$$\left|P_t^{D,V_n} f(x)\right| \leq P_t^{D,V} |f|(x)$$

and

$$\left|P_t^{D,V_n} \mathcal{A}_n f(x)\right| \leq P_t^{D,V} \left(\left|\frac{\Delta}{2}\right| + |Vf|\right)(x) \quad .$$

Hence by the dominated convergence we derive that

$$P_t^{D,V}f - f = \int_0^t P_s^{D,V}\mathcal{A}f\,ds \quad , \quad \forall t \geq 0.$$

It follows that f is in the domain of the generator $\mathcal{L}_{(\infty)}^{D,V}$ of C_0-semigroup $\left\{P_t^{D,V}\right\}_{t\geq 0}$.

Step 2: the general case. Setting $V^n = V \vee (-n)$, for $n \in \mathbb{N}$, and denoting by

$$\mathcal{A}^n = -\frac{\Delta}{2} + V^n$$

the generator of the C_0-semigroup $\left\{P_t^{D,V^n}\right\}_{t\geq 0}$ on $(L^\infty(D,dx), \mathcal{C}\left(L^\infty, L^1\right))$, we have by Step 1

$$P_t^{D,V^n}f - f = \int_0^t P_s^{D,V^n}\mathcal{A}^n f\,ds \quad , \quad t \geq 0.$$

Notice that

$$\left|P_s^{D,V^n}\mathcal{A}^n f(x)\right| \leq P_s^{D,V}\left(\left|\frac{\Delta}{2}f\right| + |Vf|\right)(x)$$

which is uniformly bounded in $L^\infty(D,dx)$ over $[0,t]$. By Fubini's theorem we have

$$\int_0^t P_s^{D,V}\left(\left|\frac{\Delta}{2}f\right| + |Vf|\right)(x)ds < \infty \text{ dx-a.e. on } D.$$

On the other hand, for any $x \in D$ fixed such that

$$P_s^{D,V}\left(\left|\frac{\Delta}{2}f\right| + |Vf|\right)(x) < \infty$$

then by dominated convergence we find

$$P_s^{D,V^n}\left(-\frac{\Delta}{2} + V^n\right)f(x) \longrightarrow P_s^{D,V}\left(-\frac{\Delta}{2} + V\right)f(x) \quad .$$

Thus by dominated convergence we have dx-a.e. on D,

$$\int_0^t P_s^{D,V^n}\left(-\frac{\Delta}{2} + V^n\right)f\,ds \to \int_0^t P_s^{D,V}\left(-\frac{\Delta}{2} + V\right)f\,ds \quad , \quad \forall t \geq 0.$$

The same argument shows that

$$P_t^{D,V^n} f - f \to P_t^{D,V} f - f \quad .$$

By consequence

$$P_t^{D,V} f - f = \int_0^t P_s^{D,V}\left(-\frac{\Delta}{2} + V\right) f ds \quad , \quad \forall t \geq 0.$$

Hence f is in the domain of generator $\mathcal{L}_{(\infty)}^{D,V}$ of semigroup $\left\{P_t^{D,V}\right\}_{t\geq 0}$. So $\mathcal{L}_{(\infty)}^{D,V}$ is an extension of the operator $(\mathcal{A}, C_0^\infty(D))$ and the lemma is proved.

Next we prove the $L^\infty(D, dx)$-uniqueness of $\mathcal{A}$. By [**WZ'06**, Theorem 2.1, p. 570], we deduce that the operator $(\mathcal{A}, C_0^\infty(D))$ is $L^\infty(D, dx)$-unique if and only if for some λ, the range $(\lambda I - \mathcal{A})(C_0^\infty(D))$ is dense in $(L^\infty(D, dx), \mathcal{C}(L^\infty, L^1))$. It is enough to show that for any $h \in L^1(D, dx)$ which satisfies the equality

$$\langle h, (\lambda I + \mathcal{A}) f \rangle = 0 \quad , \quad \forall f \in C_0^\infty(D)$$

it follows $h = 0$.

Let $h \in L^1(D, dx)$ be such that for some λ one have

$$\langle h, (\lambda I + \mathcal{A}) f \rangle = 0 \quad , \quad \forall f \in C_0^\infty(D)$$

or

$$(\lambda I + \mathcal{A}) h = 0 \quad \text{in the sense of distribution.}$$

Since $V \in L_{loc}^\infty(D, dx)$, by applying [**AS'82**, Theorem 1.5, p. 217] we can see that h is a continuous function. By the mean value theorem due to Aizenman and Simon [**AS'82**, Corollary 3.9, p. 231], there exists some constant $C > 0$ such as

$$|h(x)| \leq C \int_{|x-y|\leq 1} |h(y)|\, dy \quad , \quad \forall x \in D.$$

As $V^- \in \mathcal{K}^d$, C may be chosen independently of $x \in D$. Since $h \in L^1(D, dx)$, it follows that h is bounded and, consequently, $h \in L^2(D, dx)$. Now by the $L^2(D, dx)$-uniqueness of $(\mathcal{A}, C_0^\infty(D))$ and [**WZ'06**, Theorem 2.1, p. 570], h belongs to the domain of the generator $\mathcal{L}_{(2)}^{D,V}$ of $\left\{P_t^{D,V}\right\}_{t\geq 0}$ on L^2 and

$$\mathcal{L}_{(2)}^{D,V} h = \left(-\frac{\Delta}{2} + V\right) h = -\lambda h \quad .$$

Hence

$$P_t^{D,V} h = e^{-\lambda t} h \quad , \quad \forall t \geq 0.$$

Let

$$\lambda(D, V) := \inf_{f \in C_0^\infty(D)} \left\{ \frac{1}{2} \int_D |\nabla f|^2 dx + V f^2 dx \;:\; \|f\|_2 \leq 1 \right\}.$$

be the lowest energy of the Schrödinger operator. If we take $\lambda < \lambda(D, V)$, then the last equality is possible only for $h = 0$, because $\left\|P_t^{D,V}\right\|_2 = e^{-\lambda(D,V)t}$ (see Albeverio and Ma [**AM'91**, Theorem 4.1, p. 343]).■

Remarque 2.4. Intuitively, to have $L^1(D, dx)$-uniqueness, the repulsive potential V^+ should grow rapidly to infinity near ∂D, this means

$$(C_1) \qquad \mathbb{P}_x\left(\int_0^{\tau_D} V^+(B_s)\, ds + \tau_D = \infty \right) = 1 \quad \text{for a.e. } x \in D$$

so that a particle with starting point inside D can not reach the boundary ∂D (see [**Wu'98**, Theorem 1.1, p. 279]).

By analogy with the uniqueness in $L^1(D, dx)$, the $L^\infty(D, dx)$-uniqueness of $(\mathcal{A}, C_0^\infty(D))$ means that a particle starting from the boundary ∂D can not enter in D. Unfortunately, here we have a problem: $L^\infty(D, dx)$-uniqueness of $\mathcal{A}$ is equivalent to the existence of a unique boundary condition for $\mathcal{A}^*$. It is well known that there are many boundary conditions (Dirichlet, Newmann, etc.). Remark that in the case of $L^1(D, dx)$-uniqueness

of $\mathcal{A}$, the effect of the boundary condition for $\mathcal{A}^*$ is eliminated by the condition (C_1) for potential. To find such condition in the case of $L^\infty(D, dx)$-uniqueness is very dificult. In this moment we can present here an interesting result from [**WZ'06**]:

Proposition 2.5. *Let D be a nonempty open domain of $\mathbb{R}^d$. If the Laplacian $(\Delta, C_0^\infty(D))$ is $L^\infty(D, dx)$-unique, then $D^C = \emptyset$ or $D = \mathbb{R}^d$.*

For the heat diffusion equation we can formulate the next result

Corollary 2.6. *If $V \in L^\infty_{loc}(\mathbb{R}^d, dx)$ and $V^- \in \mathcal{K}^d$, then for every $h \in L^1(\mathbb{R}^d, dx)$, the heat diffusion equation*

$$\begin{cases} \partial_t u(t,x) = \left(-\frac{\Delta}{2} + V\right) u(t,x) \\ u(0,x) = h(x) \end{cases}$$

has one $L^1(\mathbb{R}^d, dx)$-unique weak solution which is given by $u(t,x) = P_t^V h(x)$.

Proof. The assertion follows by [**WZ'06**, Theorem 2.1, p. 570] and Theorem 2.2. ∎

Acknowledgements. I am grateful to Professor Liming WU for his kind invitation to Wuhan University, China, during May-June 2006 where this result was reported and for his valuable help and support. And I want to thank to anonymous reviewer for sugestions.

References

[**AM'91**] Albeverio, S., Ma, Z.M. Perturbation of Dirichlet form: lower boundedness, closability and form cores. *J. Funct. Anal.*, **99**(1991), 332-356.

[**AS'82**] Aizenman, M., Simon, B. Brownian motion and Harnack's inequality for Schrödinger Operators. *Comm. Pure Appl. Math.*, **35**(1982), 209-271.

[**Ar'86**] Arendt, W. The abstract Cauchy problem, special semigroups and perturbation. *One Parameter Semigroups of Positive Operators* (R. Nagel, Eds.), Lect. Notes in Math., **1184**, Springer, Berlin, 1986.

[**Da'80**] Davies, E.B. *One-parameter semigroups.* Academic Press, London, New York, Toronto, Sydney, San Francisco, 1980.

[**Dj'97**] Djellout, H. *Unicité dans L^p d'opérateurs de Nelson.* Prépublication, 1997.

[**Eb'97**] Eberle, A. *Uniquenees and non-uniqueness of singular diffusion operators.* Doctor-thesis, Bielefeld, 1997.

[**Ka'84**] Kato, T. *Perturbation theory for linear operators.* Springer Verlag, Berlin, Heidelberg, New York, 1984.

[**Ka'72**] Kato, T. Schrödinger operators with singular potentials. *Israel J. Math.*, **13**(1972), 135-148.

[**Lo'86**] Lotz, H.P. The abstract Cauchy problem, special semigroups and perturbation. *One Parameter Semigroups of Positive Operators* (R. Nagel, Eds.), Lect. Notes in Math., **1184**, Springer, Berlin, 1986.

[**RS'75**] Reed, M., Simon, B. *Methods of Modern Mathematical Physics, II, Fourier Analysis, Self-adjointness.* Academic Press, New York, 1975.

[**Rö'98**] Röckner, M. *L^p-analysis of finite and infinite dimensional diffusion operators.* Lect. Notes in Math., **1715**(1998), 65-116.

[**Sch'71**] Schechter, M. *Spectra of partial differential operators.* North-Holland, Amsterdam, 1971.

[**Si'82**] Simon, B. Schrödinger Semigroups. *Bull. Amer. Math. Soc.* (3)**7**(1982), 447-526.

[**Wu'98**] Wu, L. Uniqueness of Schrödinger Operators Restricted in a Domain. *J. Funct. Anal.*, (2)**153**(1998), 276-319.

[**Wu'99**] Wu, L. Uniqueness of Nelson's diffusions. *Probab. Theory Relat. Fields*, **114**(1999), 549-585.

[**Wu'01**] Wu, L. L^p-uniquenessof Schrödinger operators and the capacitary positive improving property. *J. Funct. Anal.*, **182**(2001), 51-80.

[**WZ'06**] Wu, L., Zhang, Y. A new topological approach to the L^∞-uniqueness of operators and L^1-uniqueness of Fokker-Planck equations. *J. Funct. Anal.*, **241**(2006), 557-610.